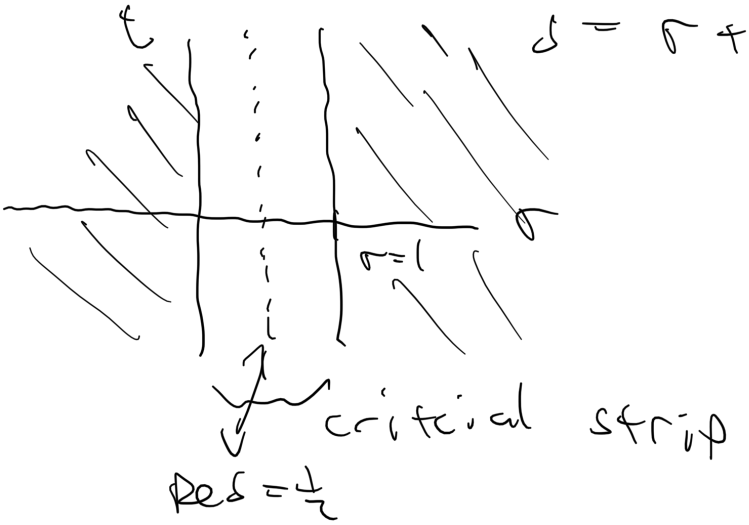
Distribution of values of L-functions

For $s \in \mathbb{C}$, $\operatorname{Re} s > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

- analytic continuation to $\mathbb{C} \setminus \{1\}$
- functional equation $s \longmapsto 1-s$

$s = \sigma + it$



$\operatorname{Re} s > 1$

$$\log \zeta(s) = - \sum_p (1 - p^{-s})$$

$$= \sum_p \frac{1}{p^s} + \cdots$$

$$\boxed{\begin{array}{c} \text{Analytic properties of } \zeta(s) \\ \Updownarrow \\ \text{Structure of } \mathbb{Z} \end{array}}$$

E.g. RH, $\zeta(s) \neq 0$, $\operatorname{Re} s > \frac{1}{2}$

$$\iff \sum_{p \leq x} 1 = \int_2^x \frac{dt}{\log t} + O(x^{1/2 + \varepsilon})$$

Problems

$\boxed{\text{i) Distribution of } \zeta(\tfrac{1}{2}+it),\ T \le t \le 2T \text{ as } T \to \infty.}$

ii) Large values of $\zeta(\frac{1}{2}+it)$

Conj $t \geq 1$ $|\zeta(\frac{1}{2}+it)| = O(t^{\varepsilon})$

iii) moments of $\zeta(\frac{1}{2}+it)$

$$M_k(T) = \frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt, \qquad k > 0.$$

arithmetic of $d_k(n) = \sum_{n_1 \cdots n_k = n} 1$

Selberg's Central limit Theorem

Thm (Selberg, 1946) Let $a \in \mathbb{R}$. Then

$$\frac{1}{T} \operatorname{meas}\left\{ t \in [T, 2T] : \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}} \le a \right\} \xrightarrow{T \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-v^2/2}\, dv$$

$T \le t \le 2T$

$$\log|\zeta(\tfrac{1}{2}+it)| \xrightarrow{\mathcal{D}} N(0, \sqrt{\tfrac{1}{2}\log\log T})$$

Sketch of Selberg's CLT

Step 1 For "most" $T \le t \le 2T$
(outside an exceptional set of size $o(T)$)

$$\boxed{\frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\ln\ln T}}} = \frac{\operatorname{Re}\sum_{p\le x}\frac{1}{p^{1/2+it}}}{\sqrt{\frac{1}{2}\ln\ln T}} + o(1).$$

$$x = T^{1/(\ln\ln T)^2}$$

Random model $\{\underbrace{p^{it}}_{e^{it\log p}}\}$

$\{\log p\}$ is linearly independent over $\mathbb{Q}$

$\Rightarrow$ behave like $\{X_p\}_p$

$X_p \leftrightarrow$ indep. ident. distr. uniform random variables on $S^1 = \{|z|=1\}$

Step 2 (easy) Let $k \in \mathbb{N}$, $x \geq 2$

$x^k \leq T^{1/3}$, Then

$$\frac{1}{T}\int_T^{2T}\Big(\mathrm{Re}\sum_{p\leq x}\frac{1}{p^{1/2+it}}\Big)^k dt$$

$$= \mathbb{E}\Big(\mathrm{Re}\sum_{p\leq x}\frac{X_p}{\sqrt{p}}\Big)^k + O(T^{-1/10})$$

complexity of

Step 3 $\mathrm{Re}\sum_{p\leq x}\frac{X_p}{p^{1/2}} \xrightarrow{\mathcal{D}} N(0, \sqrt{\tfrac{1}{2}\log\log x})$

$$\mathbb{E}\Big(\mathrm{Re}\sum_{p\leq x}\frac{X_p}{p^{1/2}}\Big)^2 = \frac{1}{4}\sum_{p,q\leq x}\frac{1}{(pq)^{1/2}}\mathbb{E}(X_p+\overline{X_p})\cdot(X_q+\overline{X_q})$$

$\mathbb{E}X_p = \mathbb{E}X_p^2 = 0$

$$= \frac{1}{2}\sum_{p\leq x}\frac{1}{p}$$

$$\sim \frac{1}{2}\int_2^x \frac{dt}{t\log t}$$

$$= \frac{1}{2}\log\log x.$$

Moments For $k>0$ let

$$M_k(T) = \frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt$$

$$= \frac{1}{T}\int_T^{2T} \exp(2k \log|\zeta(\tfrac{1}{2}+it)|)\,dt$$

$$\approx \mathbb{E}\exp(2k\, Y_T)$$

$Y_T \hookrightarrow N(0, \sqrt{\tfrac{1}{2}\log\log T})$, $Z \hookrightarrow N(0,\sigma)$

$$\mathbb{E}\, e^{tZ} = e^{t^2/2\,\sigma^2}$$

$\sigma = \sqrt{\tfrac{1}{2}\log\log T}$

$t = 2k$

$$M_k(T) \approx e^{\frac{(2k)^2}{2}(\frac{1}{2}\log\log T)}$$

$$= (\log T)^{k^2}$$

Conj For $k>0$, $\exists\, c_k>0$ s.t.

$$M_k(T) \sim c_k (\log T)^{k^2}$$

only known for $k=1,2$

central L-values

Consider $L(\frac{1}{2}, \chi)$, $\chi \bmod q$

q prime.

Q How does $L(\frac{1}{2}, \chi)$ behave as we vary $\chi \pmod q$?

<u>Conj</u> $L(\frac{1}{2}, \chi) \neq 0$ for each $\chi \bmod q$.

<u>Thm</u> (Khan – Milićević – Ngo, '22)

As $q \to \infty$

$$\#\{\chi \bmod q : L(\tfrac{1}{2}, \chi) \neq 0\} \geq \left(\frac{5}{13} + o(1)\right) \phi(q)$$

<u>Thm</u> (Bui – Evans – L. – Pratt)

Let $a \in \mathbb{R}$. Then

$$\frac{1}{\phi_w(q)} \sum_{\chi(q)} w(\chi)\, \mathbb{1}_{(-\infty, a)} \left(\frac{\log |L(\frac{1}{2}, \chi)|}{\sqrt{\frac{1}{2} \log \log q}} \right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-v^2/2}\, dv$$

$$\phi_w(q) = \sum_{\chi(q)} w(\chi)$$

$W(\chi) \approx 1_S(\chi)$, $S = \{\chi(q) : L(\tfrac{1}{2}, \chi) \neq 0\}$

" $W(\chi) = \prod (i\gamma_\chi \log q)$ "

$|\gamma_\chi| \leq \frac{10^{-5}}{\log q}$ $\leftarrow$ $L(\tfrac{1}{2} + i\gamma_\chi, \chi) = 0$

$W(\chi) \ll 1$

$|\gamma_\chi| \gg \frac{1}{(\log q)^2}$ 100% of $\chi(q)$